Contents

CHAPTER 1

Engineering Formulae

In this chapter we look at the applications of basic algebra to engineering problems.

The word *algebra* comes from the Arabic '*al-jabr*' which occurs in Al-Khwarizmi's book '*Hisab al-jabr w'al-muqabala*' written in the early ninth century. *Al-jabr* means restoration (or transpose to remove the negative quantities of an equation, e.g. $3x + 1 = 8 - 4x$ goes to $7x + 1 = 8$). Al-Khwarizmi (780–850 AD) was born in Khwarizm, now called Khiva, a town located in Uzbekistan (a former Soviet republic which became independent in 1991).

SECTION A **Substitution and transposition**

By the end of this section you will be able to:

- evaluate formulae using BROIDMAS
- solve equations
- transpose formulae

A1 **Evaluating formulae**

A **formula** is a general rule or law of mathematics. The plural of formula is **formulae**.

In evaluating formulae, the mnemonic BROIDMAS gives the order of operation (see Introductory chapter):

Brackets	} First
ROots Indices	} Second
Division Multiplication	} Third
Addition Subtraction	} Last

It's imperative that you understand BROIDMAS because it tells us the rules of algebra and is used for evaluating and simplifying algebraic expressions. Moreover it can be useful for typing in an expression into a computer algebra package or a calculator.

In algebra, letters or symbols are used to represent numbers. These letters or symbols may be **constants**, that is fixed, or **variables**, which means they can take up various values.

No space between letters represents multiplication, for example

$$ab = a \times b = a \cdot b$$

So if $a = 3$ and $b = 7$ then $ab = 3 \times 7 = 21$. To evaluate a formula we substitute the given numbers in place of letters and then apply BROIDMAS to evaluate the arithmetical expression as in the Introductory chapter.

For example, evaluate $a(b + c) + \dfrac{c(a + b)^2}{b}$ where $a = 2$, $b = 3$ and $c = 5$:

$$\begin{aligned} 2(3 + 5) + \frac{5(2 + 3)^2}{3} &= (2 \times 8) + \left(\frac{5 \times 5^2}{3}\right) \\ &= 16 + \left(\frac{5 \times 25}{3}\right) \\ &= 16 + \frac{125}{3} \\ &= 57\frac{2}{3} \end{aligned}$$

Example 1

Pythagoras theorem gives the length of the longest side, c, in terms of the other two sides of a right-angled triangle, a and b, as

$$c = \sqrt{a^2 + b^2}$$

Evaluate c for $a = 5$ and $b = 12$.

Solution

Substituting $a = 5$ and $b = 12$ into $c = \sqrt{a^2 + b^2}$ gives

$$\begin{aligned} c &= \sqrt{5^2 + 12^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} = 13 \end{aligned}$$

Hence $c = 13$.

A2 Transposition of formulae

In the formula $v = u + at$, we say v is the **subject** of the formula. If we want to make t the subject of the formula then we need to change the form to

$$t = \square$$

This process of changing the subject is called **transposition** of formulae.

When transposing we can

- ▶ add, subtract, multiply or divide by the same quantities on both sides of the formula (though we cannot divide by zero)

A3 Transposition applied to equations

An **equation** is a mathematical statement that says two expressions are equal. For example

$$x - 3 = 7$$

is an equation where x is a unknown variable. To solve this equation means we need to find the value (or values) of x so that

Left-Hand Side = Right-Hand Side

Hence we need to make x the subject of $x - 3 = 7$. **That is we need to remove the 3 on the Left-Hand Side. How?**

Add 3. We need to add 3 to both sides because we have to maintain the **balance** of the equation:

$$x - 3 + 3 = 7 + 3$$

$$x - 0 = 10$$

$$x = 10$$

In this case $x = 10$ is a **solution**, or a **root**, of the above equation. The process of finding the value of x is the same as transposition of formulae. Let's try some examples in the field of electrical principles and mechanics.

Example 2 *electrical principles*

If the voltage, V, across a resistor $R = 100\ \Omega$ is 10 volts, find the current I through the resistor, given that $V = IR$.

Solution

Substituting $V = 10$ and $R = 100$ into $V = IR$ gives

$$10 = 100I$$

What are we trying to find?

The value of I. **How do we find I?**

Divide both sides by 100:

$$\frac{100I}{100} = \frac{10}{100}$$

Cancelling the 100's on the Left-Hand Side:

$$I = \frac{10}{100} = 0.1 \text{ amp (A)}$$

The unit for current is amp and will generally be denoted by A.

As a check you can substitute $I = 0.1$ into $10 = 100I$, thus $10 = 100 \times 0.1$.

Example 3 *mechanics*

A vehicle's speed, v, is given by

$$v = 14 + 5t$$

where t is time. Find the time taken in seconds to reach a speed of 23 m/s.

Solution

Substituting $v = 23$ gives

$$14 + 5t = 23$$

We need to find *t*. How?

Subtract 14 from both sides:

$$5t = 23 - 14 = 9$$

How do we remove the 5 from the Left-Hand Side?

Divide both sides by 5:

$$\frac{5t}{5} = \frac{9}{5}$$

Cancelling 5's on the Left-Hand Side gives:

$$t = \frac{9}{5} = 1.8 \text{ s}$$

We use SI units throughout the book – see the Introductory chapter. For example, velocity is given in m/s, acceleration in m/s^2, time in s, etc.

The above equations, $14 + 5t = 23$ and $100I = 10$, are examples of **linear equations**.

A4 Transposition applied to engineering formula

Algebraic expressions can be simplified by adding, subtracting or cancelling like terms, for example

$$x + x = 2x, \quad x - x = 0, \quad x + 5x = 6x \quad \text{and} \quad \frac{x}{x} = 1 \quad [\text{provided } x \neq 0]$$

We can **only** add and subtract like terms. We cannot simplify the following:

$$x + y = x + y, x - y = x - y, x + 5y = x + 5y \quad \text{and} \quad \frac{x}{y} = \frac{x}{y}$$

The procedure in applying transposition to formulae is very similar to that used in solving equations. Let's try some engineering examples.

Example 4 *electronics*

The power, P, dissipated in a circuit is given by

$$P = IV$$

where I is current and V is voltage. Transpose the formula for V.

Solution

We want to get $V = - - -$. **How can we achieve this from $P = IV$?**

We need to remove I from the Right-Hand Side (RHS). How?

Divide both sides by I. Thus

$$\frac{P}{I} = \frac{IV}{I} \qquad \text{[the } I\text{'s on the RHS cancel out]}$$

$$\frac{P}{I} = V \text{ or } V = \frac{P}{I}$$

In Example 4, how do we know that we need to divide both sides by I?

The subject that we want to obtain is V.

What does the formula $P = IV$ do to V?

It is multiplied by I.

We want to remove the I and find V on its own. **How can we remove I?**

We can divide by I.

One way of obtaining the subject in many cases is to see what the formula does to the subject and then do the opposite on the other side. In **Example 4** the subject (V) is multiplied by I so we need to divide the other side by I, thus obtaining $V = \frac{P}{I}$.

Example 5 *mechanics*

The velocity, v, of an object with an initial velocity u and constant acceleration a after time t is given by

$$v = u + at$$

Transpose to make t the subject of the formula.

Solution

We need to get from $v = u + at$ to $t = - - -$

Example 5 *continued*

We need to remove the u first. How?

Subtract u from both sides:

$$v - u = (u + at) - u$$
$$= \underbrace{u - u}_{=0} + at$$
$$v - u = at$$

How can we obtain t from $v - u = at$?

Divide both sides by a:

$$\frac{v - u}{a} = \frac{at}{a}$$

The a's on the Right-Hand Side cancel out to give

$$t = \frac{v - u}{a}$$

We will assume that the variable we are dividing by is **not** zero because we cannot divide by zero. So in **Example 5**, a is not zero.

SUMMARY

In evaluating the formula we use the mnemonic BROIDMAS: <u>B</u>rackets, <u>RO</u>ots, <u>I</u>ndices, <u>D</u>ivision, <u>M</u>ultiplication, <u>A</u>ddition, <u>S</u>ubtraction.

We can transpose a formula to make a certain variable the subject of the formula. Transposing involves arithmetical operations carried out on both sides of the formula. We use transposition to solve equations.

Exercise 1(a)

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Given that

$$c = \sqrt{a^2 + b^2}$$

evaluate c for $a = 24$ and $b = 7$.

2 [*electrical principles*] If the voltage, V, across a resistor $R = 1000\ \Omega$ is 15 V, then find the current I given that

$$V = IR.$$

3 [*thermodynamics*] A gas is expanded from an initial pressure P_1 and volume V_1 of 5×10^6 N/m^2 and 2×10^{-4} m^3 respectively, to a final pressure $P_2 = 2 \times 10^7$ N/m^2. Find the new volume V_2 given that $P_1V_1 = P_2V_2$.

4 [*thermodynamics*] A gas has pressure $P = 5.6 \times 10^5$ N/m^2, volume $V = 0.015$ m^3 and is at a temperature $T = 312$ K. If there are $n = 34.6$ mole of gas, determine the mass, m, given that

$$PV = nmRT$$

where $R = 8.31$ J/(K mole) (R is called the universal or molar gas constant).

Exercise 1(a) continued

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

5 [*mechanics*] The distance, s, travelled in time t is related by

$$s = ut + \frac{1}{2}at^2$$

where u is the initial velocity and a is constant acceleration. Determine a, given that $s = 30$ m, $u = 2$ m/s and $t = 5$ s.

6 [*electrical principles*] The resistance, R, of a wire at $t\,°C$ is given by

$$R = R_0(1 + \alpha t)$$

where R_0 is the resistance at $0\,°C$ and α is the temperature coefficient of resistance. Determine α, given that $R_0 = 33\ \Omega$, $R = 35\ \Omega$ and $t = 89\,°C$. (The units of α are $/\,°C$.)

7 [*electrical principles*] A battery with e.m.f. $E = 12$ V and an internal resistance $r = 1\ \Omega$ is connected across a resistor $R = 20\ \Omega$. Find the voltage V across R, given that

$$E = \frac{V(R + r)}{R}$$

8 [*mechanics*] The velocity, v, of an object is given by

$$v = u + at$$

where u is the initial velocity, a is constant acceleration and t is time. Transpose to make

i u the subject

ii a the subject.

9 [*electrical principles*] Ohm's law states that

$$I = \frac{V}{R}$$

where I is current, V is voltage and R is resistance. Transpose to make R the subject of the formula.

10 [*thermodynamics*] The characteristic equation of a perfect gas is given by

$$PV = mRT$$

(P = pressure, V = volume, R = gas constant, m = mass and T = temperature). Make T the subject.

11 [*mechanics*] The slip, S (%), of a vehicle is given by

$$S = \left(1 - \frac{r\omega}{v}\right) \times 100$$

where r = radius of tyre, ω = angular velocity and v = velocity. Make ω the subject of the formula.

SECTION B **Transposing engineering formulae**

By the end of this section you will be able to:

- ▶ transpose complicated formulae
- ▶ evaluate formulae by substitution

Transposition and substitution are two very important concepts of algebra and it is critical that you fully understand the processes involved because the remaining chapters in this book rely upon these techniques.

This section is more difficult than Section A in the sense that the formulae involve roots and inverse.

B1 Formulae involving roots

As discussed in the Introductory chapter, the square root and the nth root are denoted by $\sqrt{}$ and $\sqrt[n]{}$ respectively. We can write these as

1.1 $\quad \sqrt{a} = a^{1/2}$ [square root]

1.2 $\quad \sqrt[n]{a} = a^{1/n}$ [nth root]

For example

$$\sqrt{49} = 7 \text{ (positive square root)}$$

$$\sqrt[3]{8} = 2 \text{ (because } 2 \times 2 \times 2 = 8 \quad \text{or} \quad 2^3 = 8;\ \sqrt[3]{} \text{ denotes the cube root)}$$

What is $8^{1/3}$ and $256^{1/4}$ equal to?

$$8^{1/3} = \sqrt[3]{8} = 2$$

$$256^{1/4} = 4 \text{ (because } 4 \times 4 \times 4 \times 4 = 256)$$

Now let's take a look at roots where letters represent variables.

We also have:

$$\sqrt{a^2} = \left(\sqrt{a}\right)^2 = a$$
$$\sqrt[n]{a^n} = \left(\sqrt[n]{a}\right)^n = a$$

(These can be shown by using the rules of indices which are explored in the next section.)

Example 6 *aerodynamics*

The lift force, L, on an aircraft is given by

$$L = \frac{1}{2}\rho v^2 AC$$

where ρ is density, v is speed, A is area and C is lift coefficient. Make v the subject of the formula.

Solution

How can we get $v = - - -$?

We can first find v^2 and then take the square root of both sides.

How do we get $v^2 = - - -$?

First we need to remove the $\frac{1}{2}$. How?

Multiply both sides by 2: $\quad 2L = \rho v^2 AC$

Next we need to remove ρAC from the Right-Hand Side. How?

Divide through by ρAC:

$$\frac{2L}{\rho AC} = v^2$$

Example 6 *continued*

How can we find v?

Take the square root of both sides (because $\sqrt{v^2} = v$):

$$\sqrt{v^2} = \sqrt{\frac{2L}{\rho AC}}$$

$$v = \sqrt{\frac{2L}{\rho AC}} \text{ which we may write as } \underbrace{\left(\frac{2L}{\rho AC}\right)^{1/2}}_{\text{by } \boxed{1.1}}$$

In the last line where we say 'by $\boxed{1.1}$' means the result follows by this reference quoted earlier and repeated at the bottom of this page.

We adopt this approach of quoting a reference number throughout the book and the formula itself will either be in the main text or on the bottom of the page below a horizontal line so that you do not need to flick over pages to find the reference.

Example 7 *materials*

The second moment of area, I, of a rectangle of height h and breadth b is given by

$$I = \frac{1}{12}bh^3$$

Make h the subject of the formula.

Solution

First we need to remove $\frac{1}{12}$ from the Right-Hand Side. How?

Multiply both sides by 12: $12I = bh^3$

By what means can we find h?

We can initially obtain h^3 and then find h. So divide both sides by b:

$$\frac{12I}{b} = h^3$$

and now take the cube root, $\left(\sqrt[3]{\ }\right)$, of both sides:

$$\sqrt[3]{\frac{12I}{b}} = h \left(\text{because } \sqrt[3]{h^3} = h\right)$$

or

$$h \underbrace{=}_{\text{by } \boxed{1.2}} \left(\frac{12I}{b}\right)^{1/3}$$

$\boxed{1.1}$ $\sqrt{a} = a^{1/2}$ $\boxed{1.2}$ $\sqrt[3]{a} = a^{1/3}$

As discussed in the Introductory chapter, in many engineering examples it is sufficient to give your final answer to the smallest number of significant figures consistent with the data. The intermediate working has to be one more decimal point (d.p.) or significant figure (s.f.) than is needed. Thus in order to give your final answer to 1 d.p. (or 1 s.f.) you need to work to 2 d.p. (or 2 s.f.).

In the next example we use substitution and transposition of formulae to evaluate the capacitance, C. It is more difficult than the above examples.

Example 8 *electronics*

The impedance, Z, of a circuit containing a resistor R, capacitor C and inductor L is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

where $X_L = 2\pi f L$ and $X_C = \dfrac{1}{2\pi f C}$ (f represents frequency).

Determine C if $R = 100\,\Omega$, $Z = 104\,\Omega$, $L = 0.1$ henry and $f = 50$ Hz.

Solution

Substituting $f = 50$ and $L = 0.1$ into $X_L = 2\pi f L$ gives

$$X_L = 2\pi \times 50 \times 0.1 = 10\pi$$

Substituting $X_L = 10\pi$, $Z = 104$ and $R = 100$ into the given formula, $Z = \sqrt{R^2 + (X_L - X_C)^2}$, results in

$$104 = \sqrt{100^2 + (10\pi - X_C)^2}$$

What are we trying to find?

We need to determine C but first we find X_C and then obtain C.

Squaring both sides gives

$$104^2 = 100^2 + (10\pi - X_C)^2$$

Transposing

$$104^2 - 100^2 = 816 = (10\pi - X_C)^2$$

We have

$$(10\pi - X_C)^2 = 816$$

Taking square root of both sides:

$$10\pi - X_C = \sqrt{816} = 28.566$$

Hence $X_C = 10\pi - 28.566 = 2.850$

Since $X_C = \dfrac{1}{2\pi f C}$ we have

$$\frac{1}{2\pi f C} = 2.850$$

Example 8 *continued*

Transposing

$$C = \frac{1}{2\pi f \times 2.85}$$

$$\underbrace{=}_{\substack{\text{substituting}\\ f=50}} \frac{1}{2\pi \times 50 \times 2.85} = 0.0011$$

Hence $C = 0.0011$ farad or 1.1×10^{-3} farad $= 1.1$ millifarad (mF). Remember that the prefix milli, m, represents 10^{-3}.

B2 Formulae involving the inverse

The multiplicative inverse of x ($\neq 0$) is denoted by x^{-1} and defined as

1.3 $$x^{-1} = \frac{1}{x}(= 1 \div x)$$

Example 9

Show that

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a} \quad (a \neq 0,\ b \neq 0)$$

Solution

We have

$$\left(\frac{a}{b}\right)^{-1} = \frac{1}{\left(\frac{a}{b}\right)} \quad \left[\text{by (1.3) with } x = \frac{a}{b}\right]$$

$$= 1 \div \frac{a}{b}$$

$$= 1 \times \frac{b}{a} = \frac{b}{a}$$

Remember $1 \div \frac{a}{b} = 1 \times \frac{b}{a}$ because when we divide fractions we turn the second fraction upside down and multiply.

We give this important result a reference number:

1.4 $$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

We also have $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$, but more of this in the next section.

Example 10 *thermodynamics*

A gas in a cylinder in state 1 with pressure P_1, temperature T_1 and volume V_1 expands to state 2 with pressure P_2, temperature T_2 and volume V_2. A formula relating these variables is given by $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$. Make T_1 the subject of the formula.

Solution

From $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$ we need $T_1 = - - -$. Taking the inverse, $(\ )^{-1}$, of both sides gives

$$\left(\frac{P_1V_1}{T_1}\right)^{-1} = \left(\frac{P_2V_2}{T_2}\right)^{-1}$$

$$\frac{T_1}{P_1V_1} = \frac{T_2}{P_2V_2} \quad \text{[by 1.4]}$$

How can we find $T_1 = - - -$?

Multiply both sides by P_1V_1:

$$T_1 = \frac{P_1V_1T_2}{P_2V_2}$$

SUMMARY

The square root, $\sqrt{\ }$, and the nth root, $\sqrt[n]{\ }$, are defined as

1.1 $\sqrt{a} = a^{1/2}$

1.2 $\sqrt[n]{a} = a^{1/n}$

The inverse of x ($\neq 0$) is defined as

1.3 $$x^{-1} = \frac{1}{x}$$

1.4 $$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a} \quad (a \neq 0,\ b \neq 0)$$

It is well worth spending some time learning these, 1.1 to 1.4, because they are used throughout the book.

Exercise **1(b)**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 [*electrical principles*] The power P dissipated in a resistor of resistance R is given by

$$P = \frac{V^2}{R} \quad [V \text{ is voltage}]$$

Make V the subject of the formula.

2 [*acoustics*] The speed, c, of sound in air is given by

$$c = \sqrt{\frac{\gamma P}{\rho}}$$

where γ is the specific heat ratio, P is the pressure and ρ is the density. Make P the subject of the formula.

3 [*mechanics*] The airflow over a vehicle causes drag D, which is given by

$$D = \frac{1}{2}\rho C v^2 A$$

where ρ is density, C is drag coefficient, v is velocity and A is the frontal area of the vehicle. Make A the subject of the formula.

4 [*electrical principles*] Evaluate the total resistance, R, in a circuit containing two resistors in parallel, $R_1 = 100\,\Omega$ and $R_2 = 270\,\Omega$, where

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

(Ω is the SI unit ohm used to measure electrical resistance).

5 [*mechanics*] The time, T, taken for a pendulum to make a complete swing is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where l = length of pendulum and $g = 9.81\ \text{m/s}^2$. Determine l, if $T = 0.5$ s.

6 [*electronics*] The impedance, Z, of a circuit containing a resistor R, capacitor C and inductor L is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

where $X_L = 2\pi fL$ and $X_C = \dfrac{1}{2\pi fC}$

(f represents frequency).

Determine C if $R = 50\,\Omega$, $Z = 100\,\Omega$, $L = 1$ henry and $f = 50$ Hz.

7 [*aerodynamics*] The power, P, required to drive an air screw of diameter D is given by

$$P = 2\pi k\rho n^3 D^5$$

(k = torque coefficient, ρ = density, n = number of revolutions per second). Make D the subject of the formula.

8 [*electronics*] A system with feedback β and gain A has an input voltage v_{in} given by

$$v_{in} = \left(\frac{1}{A} - \beta\right)v_{out}$$

(v_{out} = output voltage). Show that

$$\frac{v_{out}}{v_{in}} = \frac{A}{1 - A\beta}$$

9 [*materials*] A cylinder of radius r is subject to a torque T at each end, which causes it to twist. The shear stress τ is given by

$$\tau = \frac{T}{\frac{1}{2}\pi r^3}$$

Make r the subject of the formula.

10 The following formulae occur in various engineering fields. Make the letter in the square brackets the subject of the formula.

a $V = \dfrac{ER}{R + r}$ [r]

b $v^2 = u^2 + 2as$ [u]

c $v = \left(\dfrac{K + 4a/3}{\rho}\right)^{1/2}$ [K]

Exercise 1(b) continued

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

d $\eta = \dfrac{\pi P r^4 t}{8 v L}$ $[r]$

e $T = 2\pi\sqrt{\dfrac{l}{g}}$ $[l]$

f $RT = \left(P + \dfrac{a}{V^2}\right)(V - b)$ $[P]$

g $W = \dfrac{P_1V_1 - P_2V_2}{n - 1}$ $[V_1]$

h $PV^n = C$ $[V]$

i $f = C\dfrac{W}{D}\sqrt{\dfrac{h}{u}}$ $[u]$

SECTION C Indices

By the end of this section you will be able to:

- use the laws of indices to simplify expressions
- use the laws of indices in applications of thermodynamics

Do you remember what 3^5 represents?

It is $\underbrace{3 \times 3 \times 3 \times 3 \times 3}_{\text{5 copies}}$ which is equal to 243.

The 5 in 3^5 is called the **index** or the **power**. The plural of index is **indices**. In this section we will predominantly apply the rules of indices to letters rather than numbers.

The topic of indices is very important for engineers but many students do find this a difficult topic – invariably because they don't know the rules well enough.

C1 Some rules of indices

We have already stated some rules of indices in the last section, 1.1 to 1.4 . Other important rules of indices are

1.5 $a^m a^n = a^{m+n}$

1.6 $a^m \div a^n = \dfrac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$

1.7 $(a^m)^n = a^{m \times n}$

1.8 $a^0 = 1 \quad (a \neq 0)$

1.9 $a^1 = a$

1.10 $a^{-n} = \dfrac{1}{a^n} \quad (a \neq 0)$

1.11 $(ab)^n = a^n b^n$

1.12 $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n} \quad (b \neq 0)$

Example 11

Simplify the following:

a $x^3\, x^2$ **b** $\dfrac{x^3}{x^2}$ **c** $\dfrac{x}{\sqrt{x}}$ **d** $\left(\sqrt[3]{x}\right)^2 \sqrt[3]{x}$

Solution

Using the above rules we have

a $x^3\, x^2 \underbrace{=}_{\text{by } 1.5} x^{3+2} = x^5$

b $\dfrac{x^3}{x^2} \underbrace{=}_{\text{by } 1.6} x^{3-2} = x^1 \underbrace{=}_{\text{by } 1.9} x$

c $\dfrac{x}{\sqrt{x}} = \dfrac{x^1}{x^{1/2}} = \underbrace{x^{1-1/2}}_{\text{by } 1.6}$

$= x^{1/2} = \sqrt{x}$ [by 1.1]

d $\left(\sqrt[3]{x}\right)^2 \sqrt[3]{x} \underbrace{=}_{\text{by } 1.2} \left(x^{1/3}\right)^2\left(x^{1/3}\right)$

$\underbrace{=}_{\text{by } 1.7} x^{2/3}\, x^{1/3} = \underbrace{x^{(2/3)+(1/3)}}_{\text{by } 1.5} = x^1 = x$

As **Example 11** shows, the rules of indices, 1.1 to 1.12, can be used to simplify algebraic expressions. We can also apply these to show results that we have already used, such as $\sqrt{a^2} = a$:

$$\sqrt{a^2} = (a^2)^{1/2} = a^{2\times 1/2} = a^1 = a$$

Similarly we have

$$\sqrt[n]{a^n} = (a^n)^{1/n} = a^{n\times 1/n} = a^1 = a$$

Note that if $x^n = a$ then taking the nth root of both sides gives

$$(x^n)^{1/n} = a^{1/n}$$

Thus we have

† $$x = a^{1/n}$$

We call this (†) because we will refer to it later on.

Let's try an engineering example.

1.1 $\sqrt{x} = x^{1/2}$ 1.2 $\sqrt[n]{a} = a^{1/n}$ 1.5 $a^m\, a^n = a^{m+n}$

1.6 $\dfrac{a^m}{a^n} = a^{m-n}$ 1.7 $(a^m)^n = a^{m\times n}$ 1.9 $a^1 = a$

Example 12 *thermodynamics*

A gas in a cylinder is compressed according to the law

$$P_1 V_1^{1.5} = P_2 V_2^{1.5}$$

where P is pressure and V is volume. If the gas has an initial volume of $V_1 = 0.16\ \text{m}^3$ and pressure of $P_1 = 140 \times 10^3\ \text{N/m}^2$ and is then compressed to a pressure of $P_2 = 750 \times 10^3\ \text{N/m}^2$, find the new volume, V_2.

Solution

Substituting $P_1 = 140 \times 10^3$, $P_2 = 750 \times 10^3$ and $V_1 = 0.16$ into

$$P_1 V_1^{1.5} = P_2 V_2^{1.5}$$

gives

$$(140 \times 10^3) \times (0.16)^{1.5} = (750 \times 10^3) \times V_2^{1.5}$$

$$V_2^{1.5} = \frac{(140 \times 10^3) \times (0.16)^{1.5}}{750 \times 10^3} \quad \text{[Dividing by } 750 \times 10^3\text{]}$$

$$= \frac{140 \times (0.16)^{1.5}}{750} \quad \text{[Cancelling } 10^3\text{'s]}$$

$$V_2^{1.5} = 0.0119$$

Applying the index 1/1.5 to both sides and using † yields:

$$V_2 = (0.0119)^{1/1.5} = 0.052\ \text{m}^3 \text{ (2 s.f.)}$$

SUMMARY

We can use the rules of indices, 1.1 to 1.12, to simplify algebraic expressions and in engineering applications such as those in thermodynamics.

Exercise 1(c)

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Simplify the following:

a $x^5\, x^2$ **b** $x^{1/5} x^{1/2}$ **c** $\dfrac{x^3}{x^3}$ **d** $\dfrac{x^7}{x^9}$

e $\left(\sqrt[5]{x}\right)^2 \sqrt[3]{x}$

2 Simplify

a $(1 + y)^2(1 + y)$ **b** $\dfrac{(1 + x^2)^5}{(1 + x^2)^3}$

c $\left(\sqrt[3]{x^2 + x + 1}\right)^5 \sqrt[3]{x^2 + x}$

d $\left(\sqrt[3]{x^2 + x + 1}\right)^5 \sqrt[3]{x^2 + x + 1}$

Questions 3 to 5, inclusive, are on [*thermodynamics*]

3 A gas in an engine obeys the law

$$P_1 V_1^{1.45} = P_2 V_2^{1.45}$$

where P represents pressure and V represents volume.

Exercise 1(c) continued

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

If $P_1 = 2 \times 10^6$ N/m², $V_1 = 0.15$ m³ and $P_2 = 2 \times 10^5$ N/m², find V_2.

4 The work done, W, on the face of a piston by a gas is given by

$$W = \frac{C{V_2}^{-0.35} - C{V_1}^{-0.35}}{-0.35}$$

where $C = P_1 V_1^{1.35} = P_2 V_2^{1.35}$ (P and V are pressure and volume respectively and C is a constant). Show that

$$-0.35W = P_2 V_2 - P_1 V_1$$

5 The state of a gas changes from P_1, V_1 and T_1 to P_2, V_2 and T_2 (pressure, volume and temperature respectively). The characteristic equation is given by

$$\frac{P_1 V_1}{T_1} = \frac{P_2 V_2}{T_2}$$

By the polytropic law we have

$$P_1 {V_1}^n = P_2 {V_2}^n$$

By using these formulae, show that

$$\frac{T_1}{T_2} = \left(\frac{P_1}{P_2}\right)^{1-\frac{1}{n}}$$

6 [*aerodynamics*] In aerodynamics the following equation holds:

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{g/LC} \left(\frac{T_1}{T_2}\right)$$

where ρ_1, ρ_2, T_1 and T_2 represent the densities and temperatures at altitude 1 and 2 respectively. L is the rate of decrease of temperature with altitude. C is a constant and g is acceleration due to gravity. Show that

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{g-LC}{LC}}$$

SECTION D Dimensional analysis

By the end of this section you will be able to:

- apply the rules of indices to check equations which involve physical quantities

D1 Dimensional analysis

There are **three** fundamental dimensions: Mass, Length and Time (M, L and T respectively). All mechanical quantities can be expressed in terms of powers of M, L and T. (Non-mechanical quantities such as electrical current can also be expressed in terms of M, L and T, but it is easier to introduce a fourth fundamental dimension – charge Q.)

We use the following notation:

[force] represents the dimension of force

Example 13

Obtain the fundamental dimensions of velocity (units m/s), acceleration (units m/s²) and force (= mass × acceleration).

Solution

We know the units of velocity are m/s so the dimensions are

$$\frac{\text{Length}}{\text{Time}} = \frac{L}{T} = L\left(\frac{1}{T}\right) \underbrace{=}_{\text{by } 1.3} LT^{-1} \text{ etc}$$

Similarly acceleration has units m/s² so the dimensions are

$$\frac{\text{Length}}{(\text{Time})^2} = \frac{L}{T^2} = L\left(\frac{1}{T^2}\right) \underbrace{=}_{\text{by } 1.10} LT^{-2}$$

What about force?

$$\begin{aligned} \text{force} &= \text{mass} \times \text{acceleration} \\ [\text{force}] &= M \times (LT^{-2}) \\ &= MLT^{-2} \end{aligned}$$

Similarly we can evaluate the dimensions of the other quantities as shown in Table 1. Try verifying some of these in your own time.

TABLE 1

Quantity	*Units*	*Dimensions*
Area	m²	L^2
Volume	m³	L^3
Velocity	m/s	LT^{-1}
Acceleration	m/s²	LT^{-2}
Force	newton (N)	MLT^{-2}
Work (or energy)	joule (J)	ML^2T^{-2}
Power	watt (W)	ML^2T^{-3}
Pressure	N/m²	$ML^{-1}T^{-2}$
Density	kg/m³	ML^{-3}
Frequency	hertz (Hz)	T^{-1}

1.3 $\frac{1}{x} = x^{-1}$ 1.10 $\frac{1}{a^n} = a^{-n}$

Dimensional analysis is a method used in checking an equation by establishing the same dimension formula on each side of the equation, that is

$$[\text{Left-Hand Side}] = [\text{Right-Hand Side}]$$

Numbers with no units attached to them are dimensionless.

Example 14 *fluid mechanics*

Bernoulli's equation is given by

$$P + \frac{1}{2}\rho v^2 + \rho g z = \text{constant}$$

where P = pressure, ρ = density, v = velocity, z = height and g = acceleration due to gravity.

Find the dimensions of the constant.

Solution

Using Table 1 we have $\left(\text{remember } \frac{1}{2} \text{ is dimensionless}\right)$

$$\underbrace{ML^{-1}\,T^{-2}}_{P} + \underbrace{ML^{-3}}_{\rho}\underbrace{(LT^{-1})^2}_{v^2} + \underbrace{ML^{-3}}_{\rho}\underbrace{LT^{-2}}_{g}\underbrace{L}_{z}$$

$$= ML^{-1}\,T^{-2} + ML^{-3}\underbrace{L^2\,T^{-2}}_{\text{by } 1.7} + M\underbrace{L^{-3+2}}_{\text{by } 1.5}\,T^{-2}$$

$$= ML^{-1}\;T^{-2} + M\underbrace{L^{-1}}_{=L^{-3+2}}T^{-2} + ML^{-1}T^{-2}$$

Hence the constant has the dimensions $ML^{-1}T^{-2}$.

A physical requirement is that **dimensional homogeneity** holds, that is both sides of an equation have the same dimensions.

Example 15 *mechanics*

The period T of a pendulum of length l is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where g is acceleration due to gravity. Show that the formula has dimensional homogeneity.

1.5 $a^m\,a^n = a^{m+n}$

1.7 $(a^m)^n = a^{m\times n}$

Example 15 *continued*

Solution

Remember 2π is dimensionless. By Table 1, g has the dimensions LT^{-2}. So we have

$$[T] = \sqrt{\frac{L}{LT^{-2}}} = \left(\frac{\cancel{L}}{\cancel{L}T^{-2}}\right)^{\frac{1}{2}} \underbrace{=}_{\text{by } 1.12} \frac{1^{\frac{1}{2}}}{(T^{-2})^{\frac{1}{2}}} = \frac{1}{\underbrace{T^{-2\times\frac{1}{2}}}_{\text{by } 1.7}} = \frac{1}{T^{-1}} = T$$

The last step is justified by

$$\frac{1}{T^{-1}} \underbrace{=}_{\text{by } 1.3} (T^{-1})^{-1} \underbrace{=}_{\text{by } 1.7} T^{(-1)\times(-1)} = T^{1} \underbrace{=}_{\text{by } 1.9} T$$

Clearly period T has dimensions T.

SUMMARY

There are three fundamental dimensions – mass M, length L and time T. We can apply the rules of indices to check dimensional homogeneity.

Exercise 1(d)

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

All questions in this exercise belong to [*dimensional analysis*].

1 Show that the dimensions of

a pressure $\left(= \frac{\text{force}}{\text{area}}\right)$ are $ML^{-1}T^{-2}$

b density $\left(= \frac{\text{mass}}{\text{volume}}\right)$ are ML^{-3}

c momentum (= mass × velocity) are MLT^{-1}

d power (= force × velocity) are ML^2T^{-3}

e impulse (= force × time) are MLT^{-1}

f kinetic energy $\left(= \frac{1}{2} \times \text{mass} \times (\text{velocity})^2\right)$ are ML^2T^{-2}

g potential energy (= mass × acceleration × height) are ML^2T^{-2}

2 The pressure, P, at a depth d of a fluid of density ρ is given by

$$P = \rho g d \quad (g = \text{acceleration})$$

Show that the formula has dimensional homogeneity.

3 Which of the following are dimensionally correct (have dimensional homogeneity)?

a $F = mgl$ **b** $s = ut + \frac{1}{2}gt^2$

c $v^2 = u^2 + 2gs$ **d** $W = F \times v$

e $P = F \times l$

(m = mass, g = acceleration, l = length, t = time, s = distance, u and v = velocities, F = force, W = work and P = power).

1.3 $\frac{1}{x} = x^{-1}$ 1.7 $(a^m)^n = a^{m\times n}$ 1.9 $a^1 = a$ 1.12 $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Exercise **1(d) continued**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

4 The dynamic coefficient of viscosity μ (viscosity of a fluid) is found from

$$F = \frac{\mu A v}{d}$$

where v = velocity, d = distance, F = force and A = area. Find the dimensions of μ.

5 Show that the following are dimensionless parameters by checking that the dimensions of each are equal to 1:

a Reynolds Number $= \dfrac{\rho v l}{\mu}$

b Mach Number $= \dfrac{v}{c}$

c Euler Number $= \dfrac{p}{\rho v^2}$

d Froude Number $= \dfrac{v}{\sqrt{gl}}$

e Weber Number $= \dfrac{v^2 l \rho}{\sigma}$

(ρ is density, v is velocity, g is acceleration due to gravity, l is length, μ is viscosity, p is pressure, c is speed of sound and σ is surface tension whose units are N/m.)

SECTION E **Expansion of brackets**

By the end of this section you will be able to:

- ▶ expand brackets
- ▶ use expansion of brackets in engineering applications
- ▶ expand brackets of the type $(a + b)(c + d)$ using FOIL

E1 **Revision of brackets**

What does $5(x + 3)$ mean?

All the terms inside the bracket are multiplied by 5:

$$5(x + 3) = (5 \times x) + (5 \times 3)$$
$$= 5x + 15$$

Let's do a few examples.

Example 16

Multiply out the brackets of the following:

a $5(2x + 1)$ **b** $3(3x - 2)$ **c** $-(x - 1)$ **d** $-2(-x - 4)$

Solution

a $5(2x + 1) = (5 \times 2x) + (5 \times 1) = 10x + 5$
b $3(3x - 2) = (3 \times 3x) - (3 \times 2) = 9x - 6$
c Remember that 'minus times minus equals plus':

Example 16 *continued*

$$-(x-1) = -1(x-1) = (-1 \times x) - \underbrace{[1 \times (-1)]}_{=-1} = -x + \underbrace{1}_{\substack{\text{because} \\ -(-1)=1}} = 1 - x$$

The result of taking a negative sign inside a bracket is to change **all** the signs inside the bracket.

d $-2(-x-4) = [-2 \times (-x)] - \underbrace{[2 \times (-4)]}_{=-8} = 2x + 8$

Example 17

Simplify the following:

a $3(x+2) + 5(2x+3)$ **b** $(x+5) - 2(x-1)$ **c** $-(2x+3) + (2x+3)$

Solution

We add all the like terms:

a
$$\begin{aligned} 3(x+2) + 5(2x+3) &= (3x+6) + (10x+15) \\ &= \underbrace{3x + 10x}_{\text{collecting all the } x \text{ terms}} + (6+15) \\ &= 13x + 21 \end{aligned}$$

b Multiplying out the brackets gives
$$\begin{aligned} (x+5) - 2(x-1) &= (x+5) - (2 \times x) - (2 \times (-1)) \\ &= (x+5) - 2x + 2 \\ &= x - 2x + (5+2) \\ &= 7 - x \end{aligned}$$

c $-(2x+3) + (2x+3) = 0$

Example 18 *structures*

The deflection, y, at a distance x from one end of a beam of length l is given by
$$y = \frac{wx^2}{6EI}(3l - x)$$
where w is the load per unit length and EI is the flexural rigidity of the beam. Remove the brackets of this expression.

Solution

We have
$$\begin{aligned} y = \frac{wx^2}{6EI}(3l - x) &= \frac{wx^2}{6EI}3l - \frac{wx^2}{6EI}x \\ &= \frac{3wx^2\,l}{6EI} - \frac{wx^2\,x}{6EI} \\ &= \frac{wx^2\,l}{2EI} - \frac{wx^3}{6EI} \end{aligned}$$

E2 **Using FOIL**

How do we remove the brackets from an expression like $(x + 3)(x + 2)$?

Each term of the first bracket (x and 3) multiplies the second bracket ($x + 2$):

$$\begin{aligned}(x+3)(x+2) &= x(x+2) + 3(x+2)\\ &= (x \times x) + (x \times 2) + (3 \times x) + (3 \times 2)\\ &= x^2 + \underbrace{2x + 3x}_{=5x} + 6\\ &= x^2 + 5x + 6\end{aligned}$$

Another way is to use FOIL, which is a mnemonic for First, Outside, Inside and Last:

$$\begin{aligned}(x+3)(x+2) &= \underbrace{(x \times x)}_{\mathrm{F}} + \underbrace{(x \times 2)}_{\mathrm{O}} + \underbrace{(3 \times x)}_{\mathrm{I}} + \underbrace{(3 \times 2)}_{\mathrm{L}}\\ &= x^2 + 5x + 6\end{aligned}$$

Multiply

The First terms in each bracket

The Outside terms

The Inside terms

The Last terms

The process of multiplying brackets is also known as **expanding brackets**.

Example 19

Expand the following:

a $(x + 4)(x + 5)$ **b** $(x + 5)(x - 1)$ **c** $(2x + 3)(3x + 5)$ **d** $(3x - 1)(4x - 2)$

Solution

Using FOIL in each case gives

a $$\begin{aligned}(x+4)(x+5) &= \underbrace{(x \times x)}_{\mathrm{F}} + \underbrace{(x \times 5)}_{\mathrm{O}} + \underbrace{(4 \times x)}_{\mathrm{I}} + \underbrace{(4 \times 5)}_{\mathrm{L}}\\ &= x^2 + \underbrace{5x + 4x}_{=9x} + 20\\ &= x^2 + 9x + 20\end{aligned}$$

b $$\begin{aligned}(x+5)(x-1) &= \underbrace{(x \times x)}_{\mathrm{F}} + \underbrace{(x \times (-1))}_{\mathrm{O}} + \underbrace{(5 \times x)}_{\mathrm{I}} + \underbrace{(5 \times (-1))}_{\mathrm{L}}\\ &= x^2 - x + 5x - 5\\ &= x^2 + \underbrace{4x}_{=5x-x} - 5\end{aligned}$$

Example 19 *continued*

c $(2x+3)(3x+5) = \underbrace{(2x \times 3x)}_{\text{F}} + \underbrace{(2x \times 5)}_{\text{O}} + \underbrace{(3 \times 3x)}_{\text{I}} + \underbrace{(3 \times 5)}_{\text{L}}$

$$= 6x^2 + \underbrace{10x + 9x}_{=19x} + 15$$

$$= 6x^2 + 19x + 15$$

d $(3x-1)(4x-2) = \underbrace{(3x \times 4x)}_{\text{F}} + \underbrace{(3x \times (-2))}_{\text{O}} + \underbrace{((-1) \times 4x)}_{\text{I}} + \underbrace{((-1) \times (-2))}_{\text{L}}$

$$= 12x^2 \underbrace{- 6x - 4x}_{=-10x} + 2$$

$$= 12x^2 - 10x + 2$$

E3 Important expansions

Important expansions are $(a+b)^2$ and $(a-b)^2$. Let's use FOIL to expand these.

We have

$$(a+b)^2 = (a+b)(a+b) = \underbrace{(a \times a)}_{\text{F}} + \underbrace{(a \times b)}_{\text{O}} + \underbrace{(b \times a)}_{\text{I}} + \underbrace{(b \times b)}_{\text{L}}$$

$$= a^2 + \underbrace{ab + ba}_{=2ab} + b^2$$

$$= a^2 + 2ab + b^2$$

Similarly we have

$$(a-b)^2 = (a-b)(a-b) = \underbrace{(a \times a)}_{\text{F}} - \underbrace{(a \times b)}_{\text{O}} - \underbrace{(b \times a)}_{\text{I}} + \underbrace{(b \times b)}_{\text{L}}$$

$$= a^2 - ab - ba + b^2$$

$$= a^2 - 2ab + b^2$$

Note these results:

$$(a+b)^2 \neq a^2 + b^2 \quad \text{[Not equal]}$$

$$(a-b)^2 \neq a^2 - b^2 \quad \text{[Not equal]}$$

The symbol '≠' means 'does not equal'.

It is useful to remember these results:

1.13 $\quad (a+b)^2 = a^2 + 2ab + b^2$

1.14 $\quad (a-b)^2 = a^2 - 2ab + b^2$

For example, by using 1.13 with $a = 2x$ and $b = 3$ we have

$$(2x+3)^2 = (2x)^2 + (2 \times 2x \times 3) + 3^2$$

$$= 2^2x^2 + (4x \times 3) + 9$$

$$= 4x^2 + 12x + 9$$

Similarly, using 1.14 with $a = 5x$ and $b = 2$ we have

$$\begin{aligned}(5x - 2)^2 &= (5x)^2 - (2 \times 5x \times 2) + 2^2 \\ &= 5^2x^2 - (10x \times 2) + 4 \\ &= 25x^2 - 20x + 4\end{aligned}$$

Another important result which will be discussed in **Exercise 1(e)** is

1.15 $$(a - b)(a + b) = a^2 - b^2$$

Expansions of the type 1.13, 1.14 and 1.15 are prevalent in many fields of engineering and are worth learning until they become second nature to you.

Example 20 *electrical principles*

The impedance, Z, of a circuit is given by

$$Z^2 = R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2$$

where R is resistance, L is inductance, C is capacitance and ω is angular frequency. Expand the brackets and simplify.

Solution

Substituting $a = \omega L$ and $b = \dfrac{1}{\omega C}$ into $(a - b)^2 = a^2 - 2ab + b^2$ produces

$$\begin{aligned}\left(\omega L - \frac{1}{\omega C}\right)^2 &= (\omega L)^2 - \left(2 \times \cancel{\omega} L \times \frac{1}{\cancel{\omega} C}\right) + \left(\frac{1}{\omega C}\right)^2 \\ &= \omega^2 L^2 - \left(\frac{2 \times L \times 1}{C}\right) + \frac{1^2}{(\omega C)^2} \\ &= \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2} \qquad \text{[Simplifying]}\end{aligned}$$

Substituting this into the original formula gives

$$Z^2 = R^2 + \omega^2 L^2 - \frac{2L}{C} + \frac{1}{\omega^2 C^2}$$

SUMMARY

Expand brackets of the form $(a + b)(c + d)$ by using FOIL (First, Outside, Inside, Last). Important expansions are

1.13 $$(a + b)^2 = a^2 + 2ab + b^2$$

1.14 $$(a - b)^2 = a^2 - 2ab + b^2$$

1.15 $$(a + b)(a - b) = a^2 - b^2$$

Exercise 1(e)

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Multiply out the brackets and simplify:

a $2(3x + 1)$ **b** $-(2x + 1)$

c $-3(5y + 1)$ **d** $x(3x + 5)$

e $3(y - 1) - (2y + 1)$

f $x(x - 3) + x(3x + 2)$

2 [*structures*] Remove the brackets from the following and simplify:

a $y = \dfrac{w}{2EI}(Lx^3 - x^4)$

b $y = \dfrac{wx^3}{8EI}(2L - 3x)$

c $y = \dfrac{wx^2}{48EI}(3L^2 - 2x^2)$

d $y = -\dfrac{w}{12EI}\left(Lx^3 - \dfrac{x^4}{2} - \dfrac{L^3x}{2}\right)$

(L is length of beam, x is distance along the beam, EI is the flexural rigidity, y is deflection of the beam and w is the load per unit length).

3 Expand the following brackets and simplify:

a $(x + 1)(x + 2)$

b $(2x + 3)(3x + 5)$

c $(2x - 1)^2$

d $(a + b)^2 - (a - b)^2$

e $(xy + 1)^2 - x(y + 1)$

4 By expanding brackets show that

a $(x - 5)(x + 5) = x^2 - 25$

b $(2x - 3)(2x + 3) = 4x^2 - 9$

c $(9x - 7)(9x + 7) = 81x^2 - 49$

What do you notice about the above results?

In general

1.15 $$(a + b)(a - b) = a^2 - b^2$$

This is known as the difference between two squares.

5 [*electrical principles*] Expand and simplify the following:

a $(R + \omega L)(R - \omega L)$

b $\dfrac{1}{R^2} + \left(\omega C - \dfrac{1}{\omega L}\right)^2$

(R is resistance, ω is angular frequency, L is inductance and C is capacitance).

6 Find

$$(x - a)(x - b)(x - c) \ldots (x - z)$$

where ... means (x − number represented by next letter of the Roman alphabet).

SECTION F **Factorization**

By the end of this section you will be able to:

- factorize simple expressions
- factorize quadratic expressions

F1 **Factorizing expressions**

We investigated factors in the Introductory chapter. **What are the factors of 10?**

$$5 \text{ and } 2 \text{ because } 5 \times 2 = 10$$

Of course there are other factors of 10:1 and 10.

Similarly $2 \times 5 \times 7 = 70$ and we say that 2, 5 and 7 are factors of 70.

In this section we look at factors of algebraic expressions.

Example 21

Factorize $5x + 5y + 5z$.

Solution

What do you notice about $5x + 5y + 5z$?

The number 5 is common to all the terms in $5x + 5y + 5z$. We write

$$5x + 5y + 5z = 5(x + y + z)$$

and say that 5 and $x + y + z$ are **factors** of $5x + 5y + 5z$.

Factorization is the reverse process of expansion discussed in the previous section.

How do we factorize an expression like

$$\mathbf{5x - 4x^2?}$$

We know x is common in both terms because $x^2 = xx$, thus

$$5x - 4x^2 = 5x - 4xx = x(5 - 4x)$$

How do we factorize an engineering expression such as

$$y = \frac{wx^2}{EI} - \frac{wx^3}{EI}?$$

We know from the rules of indices that x^3 can be written as x^2x, so we have:

$$y = \frac{wx^2}{EI} - \frac{wx^2\,x}{EI}$$

What is common between the two terms on the Right-Hand Side?

Clearly it is $\dfrac{wx^2}{EI}$. So we can take out this common factor and write y as

$$y = \frac{wx^2}{EI}1 - \frac{wx^2}{EI}x = \frac{wx^2}{EI}(1 - x)$$

Let's do another example.

Example 22 *structures*

The deflection y of a beam of length L at distance x is given by

$$y = \frac{wx^2\,L^2}{16EI} + \frac{wx^4}{16EI}$$

where w is the load per unit length and EI is the flexural rigidity. Factorize this expression.

Example 22 *continued*

Solution

From the rules of indices we have $x^4 = x^2x^2$, so we can write y as

$$y = \frac{wx^2 L^2}{16EI} + \frac{wx^2 x^2}{16EI}$$

$\dfrac{wx^2}{16EI}$ is common to both terms on the Right-Hand Side, so we can take this factor out:

$$y = \frac{wx^2}{16EI}(L^2 + x^2)$$

The next example is a lot more difficult because it involves an algebraic fraction with different denominators.

Example 23 *structures*

The deflection, y, of a beam of length L at distance x is given by

$$y = \frac{wx^2 L^2}{8EI} - \frac{wx^4}{24EI}$$

where w is the load per unit length and EI is the flexural rigidity. Factorize this expression.

Solution

From the rules of indices we have $x^4 = x^2x^2$, so we can write y as

$$y = \frac{wx^2 L^2}{8EI} - \frac{wx^2x^2}{24EI}$$

$\dfrac{wx^2}{EI}$ is common to both terms on the Right-Hand Side, so we can take this factor out:

*

$$y = \frac{wx^2}{EI}\left(\frac{L^2}{8} - \frac{x^2}{24}\right)$$

Can we factorize this further?

Yes. The bracket term $\dfrac{L^2}{8} - \dfrac{x^2}{24}$ is an example of an algebraic fraction. It is dealt with in the same way as an arithmetic fraction.

How do you evaluate $\dfrac{1}{8} - \dfrac{1}{24}$?

We need a common denominator, 24. Hence

$$\frac{1}{8} - \frac{1}{24} = \underbrace{\frac{3}{24}}_{=1/8} - \frac{1}{24}$$

Example 23 *continued*

Similarly we have

$$\frac{L^2}{8} - \frac{x^2}{24} = \frac{3L^2}{24} - \frac{x^2}{24} = \frac{3L^2 - x^2}{24}$$

(Of course we cannot simplify $3L^2 - x^2$ any further because they are not like terms.)

Substituting $\frac{L^2}{8} - \frac{x^2}{24} = \frac{3L^2 - x^2}{24}$ into $\boxed{*}$ gives

$$\begin{aligned} y &= \frac{wx^2}{EI}\left(\frac{3L^2 - x^2}{24}\right) \\ &= \frac{wx^2}{24EI}(3L^2 - x^2) \end{aligned}$$

In **Example 23** the examination of the fraction, $\frac{1}{8} - \frac{1}{24}$, might seem like a diversion, but to deal with the algebraic fraction, $\frac{L^2}{8} - \frac{x^2}{24}$, we need to consider the arithmetic fraction.

F2 **Factorizing quadratics ($ax^2 + bx + c$)**

An expression of the form $ax^2 + bx + c$ (where a is not zero) is called a **quadratic.** Expand $(x + 2)(x + 5)$.

We can use FOIL:

$$\begin{aligned} (x + 2)(x + 5) &= \underbrace{(x \times x)}_{\text{F}} + \underbrace{(x \times 5)}_{\text{O}} + \underbrace{(2 \times x)}_{\text{I}} + \underbrace{(2 \times 5)}_{\text{L}} \\ &= x^2 + 5x + 2x + 10 \\ &= x^2 + 7x + 10 \end{aligned}$$

Remember in this section we go in the opposite direction.

How can we obtain $(x + 2)(x + 5)$ given the quadratic $x^2 + 7x + 10$? (Or how do we factorize $x^2 + 7x + 10$?)

Let's assume we don't know the factors of $x^2 + 7x + 10$. We know $x^2 + 7x \underset{\uparrow}{\underline{\pm}} 10 = (x \pm *)(x \pm \bullet)$ because $x \times x$ gives x^2.

1 If the sign in front of 10 is

+ then ± and ± in the brackets are the same sign
− then ± and ± in the brackets are different signs

In this example, ± and ± are the same sign but we have to establish which sign.

2 If the signs are the same then

$$x^2 \underset{\uparrow}{\underline{\pm}} 7x + 10 = (x \pm *)(x \pm \bullet)$$

this first sign tells you what the sign is, hence

$$x^2 + 7x + 10 = (x + *)(x + \bullet)$$

3 Now we look at the factors of 10 (because 10 is the only term in the quadratic which does not contain an x). **What are the factors of 10?**

10 and 1 or 5 and 2

We have a $7x$ on the Left-Hand Side, therefore

$$\begin{aligned} 7x &= x\bullet + {*}x && \text{[Expanding]} \\ &= (\bullet + {*})x && \text{[Factorizing]} \end{aligned}$$

Since we want 7 in the middle, * must be 5 and • must be 2 (or vice versa). So we have

$$x^2 + 7x + 10 = (x + 5)(x + 2)$$

Let's do another example.

Example 24

Factorize $x^2 - 2x - 3$.

Solution

Using the above procedure we have

$$x^2 - 2x \underset{\uparrow}{-} 3 = (x + \quad)(x - \quad)$$

Because of this, the signs are different. Next we look at the factors of 3.

What are the factors of 3?

1 and 3

Hence we have

$$(x + 1)(x - 3) \text{ or } (x + 3)(x - 1)$$

Since we want -2 in the middle it is -3 and $+1$. Thus

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

How do we factorize $x^2 + 5x - 3$?

Since the only factors of 3 are 1 and 3 we can only have

$$(x + 1)(x - 3) \text{ or } (x - 1)(x + 3)$$

Multiplying out either of these does not give

$$x^2 + 5x - 3$$

Where have we made a mistake?

There is **no** mistake. Simply, **not** all quadratics, $ax^2 + bx + c$, can be factorized into whole numbers. The actual factorization is

$$x^2 + 5x - 3 = \left(x + \frac{5 - \sqrt{37}}{2}\right)\left(x + \frac{5 + \sqrt{37}}{2}\right)$$

Of course this looks horrendous and you are not expected to attempt this factorization in this chapter. The quadratic $x^2 + 5x - 3$ cannot be factorized into simple whole numbers.

Example 25

Factorize $2x^2 + 7x - 15$.

Solution

We know:

$$2x^2 + 7x \underbrace{-}_{\uparrow} 15 = (2x \pm *)(x \pm \bullet) \qquad \text{[Because we want } 2x^2\text{]}$$

This sign tells us that the signs in the middle (± and ±) are different.

So we have

$$(2x - *)(x + \bullet) \text{ or } (2x + *)(x - \bullet)$$

Let's consider the case $(2x - *)(x + \bullet)$. The factors of 15 are 15 and 1 or 5 and 3. We need a 7 in the middle (the x term). In this example we need to be careful because the middle term is obtained by

$$(2x - *)(x + \bullet) = \ldots \overbrace{\underbrace{2x \times \bullet}_{\text{outside}} + \underbrace{x \times (-*)}_{\text{inside}}}^{\text{only the middle term}} \ldots$$

Clearly the factors 15 and 1 are useless because we will never get 7. They need to be 5 and 3 because $(2 \times 5) - 3 = 7$. So $\bullet$ is 5 and $*$ is 3, that is the x term is made from $(2 \times 5) - 3 = 7$. We have

$$2x^2 + 7x - 15 = (2x - 3)(x + 5)$$

You can always check your result; expanding $(2x - 3)(x + 5)$ gives $2x^2 + 7x - 15$. Also note that if you change the signs such that we have $(2x + 3)(x - 5)$ then you get $-7x$ in the middle and not $+7x$ as required. You can only judge the placement of signs by practising a number of factorizations. It's good practice to expand your final factorization to check your result.

F3 **Important factorization**

? How do we factorize $x^2 - 25$?

It is a quadratic because the highest power term is x^2 and it doesn't matter if there is no x. Remember a quadratic is $ax^2 + bx + c$ where a is **not** zero but b or c may be zero.

? How do we factorize this, $x^2 - 25$?

We can use

1.15 $$a^2 - b^2 = (a + b)(a - b)$$

It's easier than the above, so we have

$$x^2 - 25 = x^2 - 5^2$$
$$\underbrace{=}_{\text{by } 1.15} (x + 5)(x - 5)$$

Similarly we have

$$x^2 - 9 = (x + 3)(x - 3)$$
$$x^2 - 16 = (x + 4)(x - 4)$$
$$x^2 - 5 = x^2 - \left(\sqrt{5}\right)^2 \qquad \left[\text{Rewriting } 5 = \left(\sqrt{5}\right)^2\right]$$
$$= \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)$$

Let's investigate a more challenging example.

Example 26 *mechanics*

The equation of an object falling in air is given by

$$ma = mg - kv^2$$

where m ($\neq 0$) is the mass of the object, a is acceleration, v is velocity, g is acceleration due to gravity and k is a constant. Show that

$$a = \left(\sqrt{g} - cv\right)\left(\sqrt{g} + cv\right) \text{ where } c = \sqrt{\frac{k}{m}}$$

Solution

Dividing the initial equation, $ma = mg - kv^2$, by m gives

$$a = \frac{mg}{m} - \frac{k}{m}v^2 = g - \frac{k}{m}v^2$$

From earlier work we know that $g = \left(\sqrt{g}\right)^2$ and $\frac{k}{m} = \left(\sqrt{\frac{k}{m}}\right)^2$ so we have

$$a = \left(\sqrt{g}\right)^2 - \left(\sqrt{\frac{k}{m}}\right)^2 v^2$$
$$= \left(\sqrt{g}\right)^2 - c^2 v^2 \qquad \text{where} \quad c = \left(\sqrt{\frac{k}{m}}\right)$$
$$= \left(\sqrt{g}\right)^2 - \underbrace{(cv)^2}_{\text{by } 1.11}$$

How can we place $\left(\sqrt{g}\right)^2 - (cv)^2$ into two brackets?

Use 1.15, hence

$$a = \left(\sqrt{g}\right)^2 - (cv)^2 = \left(\sqrt{g} - cv\right)\left(\sqrt{g} + cv\right)$$

1.11 $a^n b^n = (ab)^n$ 1.15 $a^2 - b^2 = (a - b)(a + b)$

SUMMARY

To factorize $ax^2 + bx + c$ we need to look at factors of a and c and signs inside the expression. A common factorization is

1.15 $$a^2 - b^2 = (a - b)(a + b)$$

You need to know this result in both directions, that is from left to right and right to left.

Exercise 1(f)

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Factorize the following:

a $4x + 4y + 4z$ **b** $8x + 8xy$
c $2x - 4y$ **d** $3x - 2x^2$
e $x^2 - xy$

2 [*mechanics*] The following formulae occur in mechanics. Factorize each of them.

a $s = ut + \frac{1}{2}at^2$

b $F = \frac{mv_2}{t} - \frac{mv_1}{t}$

c $F = \rho A v_2 v_1 - \rho A v_1^2$

3 The surface area, S, of a cone of radius r and height h is given by

$$S = \pi r^2 + \pi r(r^2 + h^2)^{1/2}$$

Factorize this formula.

4 Factorize the following:

a $x^2 + 7x + 10$ **b** $x^2 + 5x + 4$
c $x^2 - 5x + 4$ **d** $x^2 - 4x - 12$
e $2x^2 + x - 1$ **f** $x^2 - 3x - 4$
g $21x^2 + 29x - 10$

5 [*electrical principles*] Factorize the following:

a $Z^2 - R^2$ **b** $\omega^2 L^2 - \frac{1}{\omega^2 C^2}$

(Z is impedance, R is resistance, L is inductance, C is capacitance and ω is angular frequency).

6 [*aerodynamics*] The Froude efficiency, F, of a propulsive system is given by

$$F = \frac{2(VV_s - V^2)}{V_s^2 - V^2}$$

[V_s and V are velocities].

Show that

$$F = \frac{2V}{V_s + V}$$

7 [*structures*] Factorize the following:

a $y = \frac{3wLx^2}{6EI} - \frac{wx^3}{6EI}$

b $y = \frac{wLx^3}{4EI} - \frac{3wx^4}{8EI}$

c $y = \frac{wx^4}{24EI} - \frac{wLx^3}{12EI} + \frac{wL^2x^2}{24EI}$

where y is the deflection at a distance x along a beam of length L, w is load per unit length and EI is flexural rigidity.

8 [*mechanics*] The acceleration, a, of an object in vibration is given by

$$a = g - k^2\omega^2$$

where g is acceleration due to gravity, ω is angular frequency and k is a constant.

Show that $a = \left(\sqrt{g} - k\omega\right)\left(\sqrt{g} + k\omega\right)$.

9 [*structures*] The deflection, y, of a beam of length l at a distance x from one end is given by

$$y = \frac{wx^3}{12EI} - \frac{lx^2w}{8EI} + \frac{l^2wx}{24EI}$$

where EI is flexural rigidity and w is load per unit length on the beam. Show that

$$y = \frac{wx}{24EI}(2x - l)(x - l)$$

Exercise 1(f) continued

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

10 [*electrical principles*] Show that

a If $N = \dfrac{Z_0 + \frac{1}{2}Z_1}{Z_0 - \frac{1}{2}Z_1}$ then

$$Z_1 = 2Z_0\left(\frac{N-1}{N+1}\right)$$

b If $Z_1(N-1)^2 + 2Z_0(N^2-1)$

$$= Z_1(N+1)^2$$

then $Z_1 = Z_0\left(\dfrac{N^2-1}{2N}\right)$

(Z_1, Z_0 are impedances and N is a number).

Section G **Quadratic equations**

By the end of this section you will be able to:

- solve some quadratic equations of the form $ax^2 = b$
- solve some quadratic equations by factorization
- solve all quadratic equations by formula

In Section A3 we considered linear equations. In this section we consider the different methods involved in solving quadratic equations.

A **quadratic equation** is an equation with the unknown variable to the second power. It has the form

$$ax^2 + bx + c = 0 \qquad [a \neq 0]$$

where x is the unknown variable. In **Example 27** below both equations are quadratics.

G1 **Solving quadratics using factorization**

We use the process of factorization described in the previous section to solve quadratic equations.

We know from the Introductory chapter that if the result of multiplying two numbers is zero then one of the numbers must be zero. This can be stated as:

If A and B are numbers and $A \times B = 0$ then $A = 0$ or $B = 0$.

We use this to solve various equations, for example to solve $x^2 - 2x = 0$.

Since x is common in both terms we can factorize, thus

$$x^2 - 2x = xx - 2x = 0$$

$$x(x-2) = 0 \qquad \text{[Factorizing]}$$

$$x = 0 \text{ or } x - 2 = 0$$

$$x = 0 \text{ or } x = 2$$

Example 27

Solve the following equations:

a $x^2 - x - 6 = 0$ **b** $27x^2 - 6x - 5 = 0$

Solution

a What are we trying to find?

The value(s) of x satisfying $x^2 - x - 6 = 0$. **Can we factorize $x^2 - x - 6$?**

$$x^2 - x - 6 = (x + 2)(x - 3)$$

So we have

$$(x + 2)(x - 3) = 0$$

What can we say about $(x + 2)(x - 3) = 0$?

$$(x + 2) = 0 \quad \text{or} \quad (x - 3) = 0$$

Hence we have

$$x + 2 = 0 \qquad \text{or} \qquad x - 3 = 0$$

$$x = -2 \qquad \text{or} \qquad x = 3$$

b Factorizing $27x^2 - 6x - 5$ is more difficult but it can factorized into whole numbers:

$$27x^2 - 6x - 5 = (3x + 1)(9x - 5)$$

$$(3x + 1)(9x - 5) = 0$$

which gives

$$3x + 1 = 0 \qquad \text{or} \qquad 9x - 5 = 0$$

$$3x = -1 \qquad \text{or} \qquad 9x = 5$$

$$x = \frac{-1}{3} = -\frac{1}{3} \qquad \text{or} \qquad x = \frac{5}{9}$$

Remember that **not all** quadratics **can** be factorized into simple whole numbers.

Example 28 *mechanics*

A body of mass $m = 35$ kg has kinetic energy, $KE = 3500$ joule (J). Find the speed v given that

$$KE = \frac{1}{2}mv^2$$

Solution

Substituting $m = 35$ and $KE = 3500$ into $\frac{1}{2}mv^2 = KE$ gives

$$\frac{1}{2}35v^2 = 3500$$

Example 28 *continued*

How do we find v?

We first find v^2 and then take the square root.

So we need to remove the $\frac{1}{2}$ and 35 from the Left-Hand Side. How?

Multiply both sides by 2:

$$35v^2 = 3500 \times 2 = 7000$$

Divide both sides by 35:

$$v^2 = \frac{7000}{35} = 200$$

Taking the square root of both sides and using a calculator gives

$$v = \sqrt{200} = 14.14 \text{ (2 d.p.)} = 14 \text{ m/s (2 s.f.)}$$

G2 Solving quadratics using formula

The formula for solving a quadratic equation

$$ax^2 + bx + c = 0$$

where x is a unknown variable is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.16}$$

Subsequently we will show this result in Chapter 2. (See question **8** in **Exercise 2(d)**.)

If the factorization is difficult or impossible then we use 1.16. Generally students prefer to use this formula rather than factorization even when they shouldn't, for example to solve $x^2 - 2x = 0$.

Example 29 *structures*

The bending moment, M, of a beam is given by

$$M = 0.3x^2 + 0.35x - 2.6$$

where x is the distance (in m) along a beam from one end. Find the value of x for which $M = 0$.

Solution

We have

$$0.3x^2 + 0.35x - 2.6 = 0$$

Example 29 *continued*

It is not easy to factorize this, so we use formula 1.16 to determine x. For the formula, a is the number next to x^2, b is the number next to x and c is the number without any x attached to it. Hence

$$a = 0.3, \quad b = 0.35 \quad \text{and} \quad c = -2.6$$

Substituting this, $a = 0.3$, $b = 0.35$ and $c = -2.6$, into

1.16 $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives

$$x = \frac{-0.35 \pm \sqrt{0.35^2 - (4 \times 0.3 \times (-2.6))}}{2 \times 0.3}$$

$$= \frac{-0.35 \pm \sqrt{3.243}}{0.6}$$

$$= \frac{-0.35 \pm 1.801}{0.6}$$

$$x = \frac{-0.35 + 1.801}{0.6} \quad \text{or} \quad \frac{-0.35 - 1.801}{0.6}$$

$$x = 2.42 \text{ (2 d.p.)} \quad \text{or} \quad x = -3.59 \text{ (2 d.p.)}$$

Since we cannot have a distance of -3.59 m on the beam, the bending moment $M = 0$ is at $x = 2.42$ m.

SUMMARY

For a quadratic equation, $ax^2 + bx + c = 0$, first seek factorization. If this fails then try the formula

1.16 $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise 1(g)

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Solve the following equations:

a $2x - 1 = 0$

b $x^2 + 5x + 6 = 0$

c $x^2 - 10x + 21 = 0$

d $6x^2 - 13x - 5 = 0$

e $5x^2 + 14x - 3 = 0$

2 [*mechanics*] A vehicle with velocity v and constant acceleration a is related by

$$v^2 = u^2 + 2as$$

where s is the distance and u is the initial velocity. If $a = 3.2$ m/s^2, $s = 187$ m and $v = 35$ m/s, find u.

Exercise 1(g) continued

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

3 [*structures*] The maximum deflection of a beam occurs at x satisfying

$$2x^2 - 3xL + L^2 = 0$$

where L is the length of the beam and x is the distance along the beam from one end. Find x at maximum deflection.

4 [*mechanics*] The displacement, s, of a particle is given by

$$s = 1.9t + 4.3t^2 \quad (t \geq 0)$$

where t is time. Find the time taken for a displacement of 50 m.

5 A rectangular conservatory has length l and its width is 5 m shorter then its length. Given that the area of the floor is 84 m^2, find the dimensions of the floor.

6 [*structures*] The bending moment, M, of a beam is given by

$$M = 3000 - 500x - 20x^2$$

where x is the distance along the beam from one end. At what distance is the bending moment $M = 0$.

7 [*structures*] A simply supported beam has the bending moment, M, given by

$$M = \frac{15}{8}x - \frac{29}{4}\left(x - \frac{1}{2}\right)^2$$

where x is the distance along the beam from one support. Find the value(s) of x for $M = 0$.

8 [*mechanics*] The height h (above the ground level) of a ball thrown vertically upwards is given by

$$h = -4.9t^2 + 55t + 12$$

where t is time. Find the time taken to reach the ground.

9 [*mechanics*] A ball is thrown vertically upwards from a height h_0. The height h above ground level is given by

$$h = h_0 + ut - \frac{1}{2}gt^2$$

where t is time and u is initial velocity. Find an expression of t for the ball to reach the ground.

10 [*aerodynamics*] The following equation occurs in aerodynamics:

$$\frac{-T}{2wL^{3/2}} = \frac{4kL^{5/2} - 3(DL^{1/2} + kL^{5/2})}{2L^3}$$

$$(L \neq 0,\ w \neq 0)$$

where T is thrust, w is weight, L is lift coefficient, D is drag coefficient and k ($\neq 0$) is a constant. Show that

$$L = \frac{-T \pm \sqrt{T^2 + 12kDw^2}}{2kw}$$

SECTION H **Simultaneous equations**

By the end of this section you will be able to:

▶ solve a pair of linear simultaneous equations

H1 **Solving simultaneous linear equations**

Simultaneous means occurring together. **Simultaneous equations** are a set of equations such that the unknown variables $x, y, z \ldots$ have the same values satisfying each equation. In this section we solve two simultaneous linear equations.

Example 30

Solve the simultaneous equations:

$$150x + 140y = 10.4$$

$$150x + 100y = 10$$

Solution

What are we trying to find?

The values of x and y satisfying the above equations:

† $\quad 150x + 140y = 10.4$

†† $\quad 150x + 100y = 10$

If we have one equation with one unknown then it is easy, we can apply a simple transposition of formulae. **Can we possibly get one equation with one unknown from † and †† ?**

If we subtract these equations, † – ††:

† $\quad 150x + 140y = 10.4$

†† $\quad -(150x + 100y = 10)$

we get $\quad 0 + 40y = 0.4$

Can we solve $40y = 0.4$?

This is just a **linear equation** with one unknown, y :

$$40y = 0.4$$

$$y = \frac{0.4}{40} = 0.01$$

Have we completed this problem?

No, we need to find x. **How do we find x?**

Substitute $y = 0.01$ into †† (or †):

$$150x + (100 \times 0.01) = 10$$
$$150x + 1 = 10$$
$$150x = 9$$
$$x = \frac{9}{150} = 0.06$$

Hence

$$x = 0.06 \text{ and } y = 0.01$$

Example 30 *continued*

We can check our solution by substituting $x = 0.06$ and $y = 0.01$ into the original equations:

$$150x + 140y = 10.4$$
$$150x + 100y = 10$$

We get

$$(150 \times 0.06) + (140 \times 0.01) = 10.4$$
$$(150 \times 0.06) + (100 \times 0.01) = 10$$

The procedure outlined in the above example is a process of **elimination**. We eliminate one of the unknown variables and then solve for the remaining unknown variable.

Example 31 *mechanics*

The distance, s, travelled by an object is given by

$$s = ut + \frac{1}{2}at^2$$

where a is constant acceleration, u is initial velocity and t is time.

An experiment produces the following results: After times of 3 s and 5 s the distances travelled by the object are 66 m and 160 m respectively.

Determine the values of u and a.

Solution

Substituting $t = 5$ and $s = 160$ into $ut + \frac{1}{2}at^2 = s$ gives

$$5u + \underbrace{\frac{1}{2}5^2}_{=12.5}a = 160$$

Substituting $t = 3$ and $s = 66$ into $ut + \frac{1}{2}at^2 = s$ gives

$$3u + \underbrace{\frac{1}{2}3^2}_{=4.5}a = 66$$

Rewriting these as

† $\quad 5u + 12.5a = 160$

†† $\quad 3u + 4.5a = 66$

? **How can we get one equation with one unknown from † and †† ?**

In the previous example we had the same number of x's so, when we subtracted, the x's vanished. **Can we remove the u's from † and †† ?**

Yes, we need to make the numbers in front of u (the coefficients of u) to be the same. In † we have $5u$ and in †† we have $3u$.

Example 31 *continued*

If we multiply $5u$ by 3 we get $15u$ and if we multiply $3u$ by 5 we also get $15u$. Thus multiplying † by 3 gives

$$(3 \times 5u) + (3 \times 12.5a) = 3 \times 160$$

* $$15u + 37.5a = 480$$

and multiplying †† by 5 yields

** $$15u + 22.5a = 330$$

Why?

Because when we subtract, * – **, the u's are eliminated:

$$\begin{aligned} &* & 15u + 37.5a &= 480 \\ &** & -(15u + 22.5a &= 330) \\ \hline & & 0 + 15a &= 150 \\ & & a &= \frac{150}{15} \\ & & a &= 10 \end{aligned}$$

Substituting $a = 10$ into $3u + 4.5a = 66$ gives the linear equation

$$3u + (4.5 \times 10) = 66$$

$$3u + 45 = 66$$

$$3u = 66 - 45 = 21$$

$$u = \frac{21}{3} = 7$$

We have $a = 10\ \text{m/s}^2$ and $u = 7$ m/s. You can check your result by plugging these numbers into the original equations.

Why do you think we remove the u's in the above example?

It is straightforward to find a common multiple of 3 and 5, that is 15, rather than find a common multiple of 4.5 and 12.5. We say that the 5 of $5u$ is the coefficient of u.

SUMMARY

For two simultaneous linear equations, eliminate one of the unknown variables and the result is a linear equation with one unknown. Solve for this unknown, substitute this value into one of the original equations and solve for the remaining unknown.

Exercise **1(h)**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Solve the simultaneous equations:

$$8x + 5y = 13$$
$$x + 5y = 6$$

2 [*mechanics*] A lifting machine obeys the law

$$E = aW + b$$

where E is effort force, W is load and a, b are constants. An experiment produces the following results: Effort forces of 45.5 N and 53 N lift loads of 70 N and 120 N respectively. Find the values of the constants a and b.

3 [*mechanics*] The displacement, s, of a body is given by

$$s = ut + \frac{1}{2}at^2$$

where t is time, u is initial velocity and a is acceleration. If at $t = 2$ s then $s = 33$ m and at $t = 3$ s then $s = 64.5$ m, find the initial velocity (u) and acceleration (a).

4 [*electrical principles*] By applying Kirchhoff's law in a circuit we obtain

$$25(I_1 - I_2) + 56I_1 = 2.225$$
$$17I_2 - 3(I_1 - I_2) = 1.31$$

where I_1 and I_2 represent currents. Find I_1 and I_2.

5 [*materials*] The length, ℓ, of an alloy varies with temperature t according to the law

$$\ell = \ell_0(1 + \alpha t)$$

where ℓ_0 is the original length of the alloy and α is the coefficient of linear expansion. An experiment produces the following results:

At $t = 55\,°\text{C}$ $\quad \ell = 20.11\,\text{m}$

At $t = 120\,°\text{C}$ $\quad \ell = 20.24\,\text{m}$

Determine ℓ_0 and α. (The units of α are /°C.)

(*Hint*: Eliminate ℓ_0 by division.)

6 [*electrical principles*] Resistors R_1 and R_2 are parallel in a circuit and satisfy

$$\frac{1}{R_1} + \frac{1}{R_2} = 1.2 \times 10^{-3}$$

$$\frac{5}{R_1} + \frac{8}{R_2} = 6.6 \times 10^{-3}$$

Determine R_1 and R_2.

(*Hint*: Work in terms of $1/R_1$ and $1/R_2$.)

7 [*dimensional analysis*] The force, F, of a jet is a function of density ρ, area A and velocity v. By assuming

$$F = K\rho^a A^b v^c$$

and dimensional homogeneity, find a, b and c and express F in terms of ρ, A and v. (K, a, b and c are real numbers. *Hint*: Use the fact that the equations must be dimensionally homogeneous to write three simultaneous equations by using Table 1.)

8 [*dimensional analysis*] The power, P, required to drive an air screw depends on the diameter D, the number n of revolutions per second and density ρ. Assume

$$P = K\rho^a n^b D^c$$

where K, a, b and c are real numbers. Using dimensional analysis, or otherwise, determine a, b and c and write down the equation relating P, ρ, n and D. (Take the dimensions of n to be T^{-1}.)

Miscellaneous exercise 1

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 [*aerodynamics*] The pressure coefficient C is defined by

$$C = \frac{\frac{1}{2}\rho(v^2 - u^2)}{\frac{1}{2}\rho v^2}$$

where u, v are velocities and ρ is density. Simplify this formula.

2 [*fluid mechanics*] The pressures P_1 and P_2 at depths d_1 and d_2 respectively are given by

$$P_1 = \rho g(d - d_1)$$
$$P_2 = \rho g(d - d_2)$$

where d is depth of the fluid, ρ is the density of fluid and g is acceleration due to gravity. Show that

$$P_2 - P_1 = -\rho g(d_2 - d_1)$$

3 [*fluid mechanics*] The head loss, h, of a fluid in a pipe is given by

$$h = \frac{v_2}{g}(v_2 - v_1) - \frac{v_2^2 - v_1^2}{2g}$$

(g is acceleration due to gravity and v_1, v_2 are velocities of fluid). Show that

$$h = \frac{(v_1 - v_2)^2}{2g}$$

4 Evaluate $x^2 + x + 41$ for $x = 0, 1, 2, 3, 4$ and 5. What do your results have in common?

5 [*electronics*] The resonant frequency, f_0, of a tuned circuit is given by

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

a Evaluate f_0 for $L = 5 \times 10^{-3}$ henry and $C = 1 \times 10^{-6}$ farad.

b If $f_0 = 1000$ Hz and $L = 1 \times 10^{-3}$ henry then determine C.

6 [*electrical principles*] The total resistance, R, of two resistors, R_1 and R_2, in parallel is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Show that

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

7 [*electrical principles*] Find the total resistance, R, of a circuit consisting of three resistors, $R_1 = 10\ \text{k}\Omega$, $R_2 = 15\ \text{k}\Omega$ and $R_3 = 1.2\ \text{k}\Omega$, connected in parallel. (The total resistance, R, is given by

$$\left.\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.\right)$$

8 [*aerodynamics*] An aircraft's drag, D, at speed v in a medium of density ρ is given by

$$D = \frac{1}{2}\rho v^2 AC_D + \frac{1}{2}\rho v^2 AkC_L^2$$

where C_D, C_L are drag coefficients, A is area and k is a constant. Transpose to make v the subject of the formula.

9 [*mechanics*] The excess energy, E, of an engine between the points of maximum speed, v_1, and minimum speed, v_2, is given by

$$E = \frac{1}{2}Iv_1^2 - \frac{1}{2}Iv_2^2$$

where I is the moment of inertia. Make I the subject of the formula.

10 [*fluid mechanics*] The flow of liquid from location 1 to location 2 can be described by Bernoulli's equation:

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + h_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + h_2$$

Miscellaneous exercise 1 continued

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where v is flow velocity, p is pressure, h is height, g is acceleration due to gravity and ρ is density. Make v_1 the subject of the formula.

11 Solve the following equations:

a $5x - 1 = 0$ **b** $3x + 2 = 8$

c $(x - 1)(x + 2) = 0$

d $(3x - 1)(2x + 3) = 0$

12 [*mechanics*] The vertical displacement, y, of a projectile in motion is given by

$$y = ut - \frac{1}{2}gt^2$$

where u is the initial velocity of the projectile. Find t for $y = 10$ m and $u = 14$ m/s. (Take $g = 9.8$ m/s².)

13 [*thermodynamics*] The exit velocity, u, of a fluid from a nozzle is given by

$$u = \left\{\frac{2\gamma P_1V_1}{\gamma - 1}\left[1 - \frac{P_2V_2}{P_1V_1}\right]\right\}^{\frac{1}{2}}$$

where P_1, V_1 represent the entrance pressure and specific volume respectively and P_2, V_2 represent the exit pressure and specific volume respectively. γ is the ratio of specific heat capacities. Given that

$$P_1V_1^\gamma = P_2V_2^\gamma$$

show that

$$u^2 = \frac{2\gamma P_1V_1}{\gamma - 1}\left[1 - \left(\frac{P_2}{P_1}\right)^{1-1/\gamma}\right]$$

Find u (correct to 1 d.p.) given that

$\gamma = 1.39$, $P_1 = 5.2 \times 10^6$ N/m²,
$V_1 = 3.1 \times 10^{-3}$ m³/kg and
$V_2 = 5 \times 10^{-3}$ m³/kg.

14 [*electrical principles*] In an electrical circuit a resistor R satisfies

$$R = 1 + \frac{3(9 + R)}{12 + R}$$

Determine R.

15 Solve the following equations:

a $x^2 - 7x + 10 = 0$

b $x^2 - 1 = 0$

c $2x^2 - 3x + 1 = 0$

d $15x^2 - x - 2 = 0$

e $-100x^2 + 400x - 300 = 0$

16 [*dimensional analysis*] The following coefficients occur in aerodynamics:

$$\text{Lift coefficient} \quad C_L = \frac{L}{PA}$$

$$\text{Drag coefficient} \quad C_D = \frac{D}{PA}$$

$$\text{Moment coefficient} \quad C_M = \frac{M}{PA\ell}$$

where L is the lift (in N), D is the drag (in N), P is the pressure, A is the area, M is the moment (in Nm) and l is the length. Show that C_L, C_D and C_M are dimensionless.

17 [*structures*] The deflection, y, of a beam of length L is given by

$$y = \frac{wx^4}{36EI} - \frac{wLx^3}{8EI} + \frac{wL^4}{36EI}$$

where w is the load per unit length, EI is the flexural rigidity and x is the distance along the beam from one end. Factorize this expression.

18 [*structures*] The critical load, P, of a steel column can be obtained from

$$L\sqrt{\frac{P}{EI}} = n\pi$$

where L is the length, EI is flexural rigidity and n is a positive whole number.

Miscellaneous exercise **1 continued**

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i Transpose to make P the subject of the formula.

ii Determine P (correct to 2 d.p.) for $n = 1$, $E = 0.2 \times 10^{12}$ N/m^2, $I = 6.95 \times 10^{-6}$ m^4 and $L = 1.07$ m.

19 Solve the following equations:

a $x^2 + 3x + 1 = 0$

b $x^2 + 4x + 2 = 0$

c $5x^2 + 2x - 1 = 0$

d $1 - 3x - 2x^2 = 0$

20 [*vibrations*] A constant, C, in a vibrational problem is defined as

$$C = \frac{F_0}{k - m\alpha^2} \qquad \left(\alpha \neq \sqrt{k/m}\right)$$

where F_0 is the magnitude of the forcing function, k is the spring stiffness, m is the mass and α is the angular frequency.

If $\omega = \sqrt{\dfrac{k}{m}}$ and $r = \dfrac{\alpha}{\omega}$ then show that

$$C = \frac{F_0/k}{1 - r^2}$$

21 [*electrical principles*] Applying Kirchhoff's law to a circuit gives

$$12(I_1 + I_2) + 67I_2 = 5.794$$
$$3I_1 - 5(I_1 - I_2) = 0.306$$

where I_1 and I_2 represent currents. Determine I_1 and I_2.

22 [*fluid mechanics*] The acoustic velocity, v, is given by

$$v = (\gamma k \rho^{\gamma - 1})^{1/2}$$

Using $k\rho^\gamma = P$ and $\dfrac{P}{\rho} = RT$, show that

$$v = \sqrt{\gamma RT}$$

(k, R are constants, T is temperature, P is pressure, ρ is density and γ is the specific heat capacity ratio).

23 [*fluid mechanics*] The ratio of depths $\dfrac{d_2}{d_1}\left(\dfrac{\text{upstream depth}}{\text{downstream depth}}\right)$ of water flowing through a channel can be derived from

$$d_1^2 - d_2^2 = \frac{2Q}{g}\left(\frac{Q}{d_2} - \frac{Q}{d_1}\right)$$
$$(d_1 \neq 0,\ d_2 \neq 0,\ d_1 \neq d_2)$$

where Q is the flow rate and g is acceleration due to gravity. Given that the Froude number, F, is defined as

$$F = \frac{Q}{\sqrt{g d_1^3}}$$

show that

$$\frac{d_2}{d_1} = \frac{1}{2}\left[-1 \pm \sqrt{1 + 8F^2}\right]$$

24 Solve the following simultaneous equations:

a $2x + 3y = 5$
$x + 2y = 3$

b $3x + 8y = -18$
$5x + 5y = 25$

c $3x + 2y = 7$
$x + 5y = 6$

25 [*vibrations*] The natural frequency, ω, of a flywheel is given by

$$\omega^2 = \frac{JG}{IL}$$

where I and J are moments of inertia, G is shear modulus of elasticity and L is length.

If mass m is placed at a distance r from the centre then the natural frequency, α, of the flywheel becomes

$$\alpha^2 = \frac{JG}{(I + 2mr^2)L}$$

From these two formulae, show that

$$I = \frac{2mr^2\alpha^2}{\omega^2 - \alpha^2}$$

Miscellaneous exercise 1 continued

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

26 [*vibrations*] When looking at vibrational problems, we often need to solve the quadratic equation

$$mx^2 + \zeta x + k = 0 \quad (*)$$

where m is the mass, ζ is the damping coefficient and k is the spring constant. Show that

$$x = \frac{\zeta}{2m}\left(-1 \pm \sqrt{1 - \frac{4mk}{\zeta^2}}\right)$$

For critical damping, we need $b^2 - 4ac = 0$ of the quadratic $ax^2 + bx + c = 0$. For what value of $\zeta\ (> 0)$ in (*) does critical damping occur.

27 [*thermodynamics*] The specific heat at constant volume c_v and the specific heat at constant pressure c_p are related by

$$c_p - c_v = R$$

where R is the gas constant. If $k = \frac{c_p}{c_v}$, show that

a $c_v = \dfrac{R}{k-1}$ **b** $c_p = \dfrac{Rk}{k-1}$

28 [*vibrations*] The four natural frequencies, ω_1, ω_2, ω_3 and ω_4, of a system are given by the roots of the equation

$$\omega^4 - 402\omega^2 + 800 = 0$$

Determine ω_1, ω_2, ω_3 and ω_4.

29 [*vibrations*] The natural frequencies, ω_1, ω_2, ω_3 and ω_4, of two pendulums connected by a spring can be obtained from the equation

$$\omega^4 - 2\left(\frac{g}{\ell} + \frac{kx^2}{m\ell^2}\right)\omega^2 + \frac{g^2}{\ell^2} + \frac{2kx^2 g}{m\ell^3} = 0$$

where ℓ, x are lengths, m is mass, k is spring stiffness and g is acceleration due to gravity. Show that

$$\omega_1 = \sqrt{\frac{g}{\ell}}, \quad \omega_2 = -\omega_1,$$

$$\omega_3 = \sqrt{\frac{g}{\ell} + \frac{2kx^2}{m\ell^2}} \text{ and } \omega_4 = -\omega_3$$

30 [*aerodynamics*] Minimum drag occurs when the lift coefficient, L, satisfies

$$kL^2 = Z$$

where Z is the zero lift coefficient and k is a constant. The velocity, v, of an aircraft satisfies

$$w = \frac{1}{2}\rho v^2 LA$$

where w is weight, ρ is density and A is area. Show that

$$v^4 = \left(\frac{2w}{\rho A}\right)^2 \cdot \frac{k}{Z}$$

31 [*structures*] The deflection, y, of a beam of length, L, is given by

$$EIy = \frac{w(L-x)L}{6} - \frac{w}{6L}(L-x)^3$$

$$[EI \neq 0,\ w \neq 0]$$

where w is the load per unit length, EI is the flexural rigidity and x is the distance along the beam. Determine the value(s) of x for which the deflection is zero.

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